A comparison between buoyant vortex rings and vortex pairs

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In this paper it is shown how earlier results for buoyant vortex rings may be extended to describe the corresponding two-dimensional case, which arises in the theory of bent-over plumes. It is again assumed that in uniform surroundings the circulation remains constant while the buoyancy acts to increase the momentum of the pair. The behaviour in two dimensions is quite different from that in three, however; a buoyant vortex ring spreads linearly with height, whereas a buoyant pair spreads exponentially with height, or linearly with time (and therefore, in a bent-over plume, linearly with distance downwind).

The theory has been extended to describe the rise of buoyant rings and pairs through stably stratified surroundings having a linear density gradient. The behaviour near the maximum height reached is found to depend critically in both cases on the relative rates at which the circulation and the momentum fall to zero. If these reach zero together, the rings or pairs will steadily increase in size and come to rest at a finite height and with a finite radius. If the circulation is non-zero when the momentum vanishes, the radius begins to decrease soon after the buoyancy becomes zero, and the vortices will therefore tend to break up suddenly and mix into their surroundings. There is a considerable increase in the final height which should be attained by vortex rings or bent-over plumes if the initial circulation is increased; it is suggested that releasing smoke intermittently, rather than continuously, at high velocity might be a means of increasing the effective height of chimneys in calm conditions. When the circulation reaches zero before the momentum does, the solutions indicate that the radius becomes very large near the level of zero buoyancy.

1. Introduction

In a previous paper (Morton, Taylor & Turner (1956), which will be referred to hereafter as I), the properties of an isolated mass of buoyant fluid released from rest in stratified surroundings were investigated, with the object of obtaining results which could be applied to the atmosphere. It was later shown (Turner (1957), to be referred to as II) how the motion in such a 'buoyant cloud' can be considered as a particular case of a buoyant vortex ring. The more general theory deals with the situation in which an arbitrary amount of vorticity is injected into the ring as it is formed, and the behaviour is found to be altered greatly by this extra vorticity. The buoyant cloud corresponds to the case in which the only circulation is produced by the buoyancy, during the initial acceleration from rest; thereafter the cloud spreads linearly with height in uniform surroundings, but at a larger angle than a buoyant ring containing the same total buoyancy and more vorticity.

At the time the above work was done, there was no reason to suppose that the corresponding two-dimensional problem would have a real physical significance, although the line source of buoyancy released continuously has been treated in the literature (e.g. Rouse, Yih & Humphreys 1952). Recently, however, Scorer (1958) has followed up his earlier work on 'thermals' (the name he has adopted for what we have called 'buoyant clouds') by suggesting that the behaviour of plumes of smoke when they have been bent over by a cross-wind and become nearly horizontal, can conveniently be discussed in terms of a line source of buoyancy. It is in fact sometimes observed that a plume of smoke bent over in this way tends to split sideways into two concentrated regions with a clear space between them, and the effect is clearly shown in the photograph obtained in a laboratory channel (figure 1, plate 1), which will be discussed in more detail in §2.1. The flow in planes perpendicular to the axis of the plume is very like that in a vortex pair, with a region of greater than the mean velocity of rise in the centre and a slower region on each side. The behaviour of a short length of such a plume as it is swept downstream should be similar to that from a two-dimensional source developing in time, and Scorer has initiated experiments to study this simplified problem in the laboratory.

In this paper it will be shown that an extension of the ideas developed for vortex rings may again be used to obtain an understanding of the two-dimensional case. There are, however, certain important differences, both when one considers the generation of the motion and the subsequent rate of spread. The first purpose of this paper will therefore be to derive corresponding results to those obtained for buoyant vortex rings in uniform surroundings, and to point out where the two cases differ physically. The attempts at extension to vortex pairs in a stratified ambient fluid have also led to a more complete understanding of the vortex rings in this case, and in the second part of this paper we shall discuss the different types of behaviour which can be obtained, particularly near the maximum height reached by the rings or pairs.

2. Motion in uniform surroundings

2.1. *Buoyant vortex rings*

The assumptions and results of II will be summarized in this section, but some familiarity with the earlier papers will be assumed when comparisons are made.

The basic assumption is that in uniform surroundings, the circulation K in a buoyant vortex ring remains constant, while the buoyancy acts to increase the momentum of the mass of fluid moving with the ring. This idea may be justified as follows. It is clear that an increase of momentum implies an increase in the radius of the ring, and therefore the addition of external fluid. It is observed that at least some (and sometimes all) of the added fluid is drawn up the centre from behind, thereby tending to concentrate the buoyant fluid in a ring, even in cases where it has been uniformly distributed to start with. Thus a circuit may be taken round the vorticity-containing region which lies completely in irrotational fluid of constant density; that is, K must be constant. It is also implied that any turbulent motion is much less important than the large-scale circulation.

If we neglect the effect of small density differences in the inertia terms, the momentum P is given by

$$P = \pi\rho K R^2, \tag{1}$$

where R is the 'mean radius' of the ring. Hence the equation balancing the buoyancy force with the rate of change of momentum is

$$\frac{dP}{dt} = \pi\rho K \frac{dR^2}{dt} = gW(\rho-\rho')$$

or

$$\frac{dR^2}{dt} = \frac{F}{\pi K}. \tag{2}$$

Here the total buoyancy ρF is defined by

$$F = \frac{gW(\rho-\rho')}{\rho}, \tag{3}$$

where W is the volume of buoyant fluid, of density ρ', and ρ is the density of the surroundings. In uniform surroundings F is constant, and equation (2) may be integrated to give

$$R^2 - R_0^2 = \frac{Ft}{\pi K}. \tag{4}$$

In order to proceed further we need to make some assumption about the vorticity distribution in the moving mass, since the velocity V of a vortex ring is critically dependent on the size of the vorticity-containing region. If we assume that the distribution remains *similar* at all heights, we have

and hence

$$\left.\begin{aligned} V &= cK/R, \\ R &= \alpha x = \frac{F}{2\pi c K^2} x, \end{aligned}\right\} \tag{5}$$

where x is the height above a virtual source and c and α are constants. In terms of the distance h between states R_0 and R, time t apart, we may obtain the form

$$t = \left(\frac{h^2}{4\pi c^2 K^3}\right) F + \frac{hR_0}{cK}. \tag{6}$$

Thus, for a given circulation, *increasing* the buoyancy gives a *lower* velocity, i.e. a longer time to reach a given height.

2.2. *Buoyant vortex pairs*

When there is no density difference in the fluid through which a vortex pair is travelling, then we know (Lamb 1932, §155) that the velocity V and the momentum per unit length P' are given by

$$V = \frac{K}{4\pi R}, \tag{7}$$

and

$$P' = 2\rho K R, \tag{8}$$

where $2R$ is the separation of the pair and K is the circulation round one line vortex. The (irrotational) fluid motion accompanying the pair has an oval cross-section with semi-axes $2{\cdot}09R$ and $1{\cdot}73R$; this shape, unlike the corresponding one for the vortex ring, is independent of the size of the region containing the vorticity, provided this is small compared with R. The velocity is also independent of this distribution, a fact which makes the two-dimensional analysis possible with one assumption less than was required in three dimensions. (With widely distributed vorticity, one might of course have to make some similarity assumption, and use a different constant in equation (7).)

We may again make the assertion that the circulation K is constant, and for the same reasons; an increase in momentum implies a separation of the pair. Suppose that the pair contains a volume Q per unit length of fluid of density ρ'. Then, neglecting density differences except in the buoyancy term, we have

$$\frac{dP'}{dt} = 2\rho K\frac{dR}{dt} = gQ(\rho-\rho')$$

or

$$\frac{dR}{dt} = \frac{F'}{2K}, \tag{9}$$

where the buoyancy $\rho F'$ per unit length is defined by

$$F' = \frac{gQ(\rho-\rho')}{\rho}. \tag{10}$$

In uniform surroundings F' is constant, and (9) may be integrated to give

$$R-R_0 = F't/2K, \tag{11}$$

where $2R_0$ and $2R$ are the initial and final separations, and t is the elapsed time. The increase of radius is *linear* with time, or, referring to the suggested application to the bent-over plume, linear with distance downwind; the corresponding theoretical and experimental result in three dimensions (equation (4)) is that R^2 increases linearly with time.

At this point it is interesting to refer to figure 1 (plate 1), which shows a plan view of a bent-over plume in a 10 × 5 cm Perspex water channel. In this case the velocity of the water was about 3 cm/sec and the initial density difference of the order of $\frac{1}{10}\%$. This photograph was produced for illustration only, and should not be examined quantitatively, since the velocity across the channel was not uniform; but the splitting is clearly shown, and the spread appears to be nearly linear with distance.

In the two-dimensional case, using (7) and (9) with *no* extra assumption, one arrives at the following explicit expression for radius in terms of distance:

$$\frac{R}{R_0} = \exp\left(\frac{2\pi F'}{K^2}h\right), \tag{12}$$

where h is the height between states R_0 and R. This exponential increase of separation with height is very different from the linear spread of vortex rings predicted by equation (5).

It also follows that in the two-dimensional case,

$$t = \frac{2KR_0}{F'}\left[\exp\left(\frac{2\pi F'}{K^2}h\right)-1\right]. \tag{13}$$

This shows that, again, increasing the buoyancy for a given circulation reduces the rate of rise, because the pair spreads faster. An obvious extension of (13) leads to a prediction of the path of the centre of the plume as it is swept downwind at a known velocity; this can of course only be applied while the motion is dominated by the buoyancy and not by the turbulence in the surroundings.

Using (11) and (12), one could determine F'/K and F'/K^2 experimentally and hence in principle find F' and K. If F' is known (and in the laboratory the volume and density of fluid released usually would be measured), then one relation is sufficient to determine K.

2.3. *The generation of the circulation*

So far it has been assumed that F' and K may be specified separately, which is so when extra momentum is given to the fluid initially. The most important case, however, is that in which a mass of buoyant fluid is released from rest, and the only circulation is that generated soon after the release by the action of the buoyancy before non-buoyant fluid has been drawn up the centre. This is what in three dimensions we have called a 'buoyant cloud', and it is in the comparison of this with the corresponding case for the line source that the contrast between the two- and three-dimensional solutions may most clearly be brought out.

In three dimensions, the mean half-angle of spread for the cloud was found in II to be $\alpha \doteqdot 0{\cdot}18$, in agreement with Woodward (1959), and corresponding to particular values of F/K^2 and c in (5). Such a relation is to be expected on dimensional grounds, since F and K^2 have the same physical dimensions, and it is not easy to see how the scale could matter, whatever the exact mechanism of generation of vorticity might be. In two dimensions, on the other hand, a non-dimensional constant cannot be formed from F' and K alone, since F' is the buoyancy *per unit length*. Another length must enter the problem, and one might suggest the dimensionally correct form

$$\frac{F'R_0}{K^2} = \text{const.} = \frac{d}{2\pi}, \tag{14}$$

where $2R_0$ is the separation of the vortex pair just after the circulation has become established, as the first relation to try when interpreting observations. With this assumption, it follows from (12) that the shape of the region swept out by the pair as it rises will always be the same when it is made non-dimensional with the initial separation:

i.e.
$$\frac{R}{R_0} = \exp\left(\frac{dh}{R_0}\right), \tag{15}$$

where d, the same constant as in (14), should be obtainable by experiment. It should be noted, however, that two-dimensional experiments will be much more sensitive to the presence of walls in a laboratory tank; the rate of spread is likely to be underestimated and the region swept out will appear more nearly wedge shaped if $2R$ approaches half the total width of the tank.

The dimensional necessity for the appearance of a length in the equations together with F' and K also helps us to understand why the spread in two dimensions is not linear with height, as it was in three dimensions. Since $F'h/K^2$ is non-dimensional, all we can say using purely dimensional reasoning is that the radius must be some function of this group, multiplied by the initial radius. The nature of this functional dependence must be determined by a complete solution such as we have given; whereas in three dimensions, since F/K^2 is the relevant non-dimensional group and this does not involve a length, we are led immediately to the result that R varies linearly with h.

3. Motion in a stably stratified fluid with constant density gradient

3.1. *Buoyant vortex rings*

Provided the analysis is restricted to the case where the extreme (potential) density differences over the region considered are small compared to some chosen reference density, say ρ_1 at the height of the source, then many of the relations obtained for a uniform fluid are again valid for the rise of a buoyant vortex ring through a stratified fluid which is at rest. We have

$$\left.\begin{aligned} P = \pi\rho_1 KR^2, \quad \frac{dP}{dt} = \rho_1 F \\ \text{and} \qquad V = \frac{dx}{dt} = \frac{cK}{R}. \end{aligned}\right\} \tag{16}$$

An additional equation, obtained in II, describes the conservation of density deficiency; for a constant stable density gradient defined by $G = -(g/\rho_1)(d\rho/dx)$, it may be put in the form

$$\frac{dF}{dt} = -qGR^3\frac{dx}{dt}, \tag{17}$$

where q is a constant which will depend on the *shape* of the volume moving with the ring.

Some general remarks based on dimensional reasoning were made in II about the dependence of the final height on the initial buoyancy and circulation F_0 and K_0 and on G, but no solution of (16) and (17) was attempted. In I, Morton obtained a solution which, although it was based on slightly different reasoning and does not introduced K explicitly, corresponds to a particular relation between F_0 and K_0^2. This point will be brought out more clearly in the following, where we obtain the more general solution.

In order to do this, we require an equation describing the variation of K. The vorticity equation in a medium of varying density becomes

$$\frac{dK}{dt} = -\oint\frac{dp}{\rho},$$

where the integration is taken round a circuit containing one of the line vortices. Let us suppose that at any instant the fluid up the centre of the ring over a depth comparable with R has been drawn from a denser level distant sR below the ring (where s is a constant), and that the density of the surroundings has not been changed by the passage of the ring. The vorticity equation then becomes

$$\frac{dK}{dt} = -\frac{\Delta\rho}{\rho} gR = -sR^2G. \tag{18}$$

That this is a reasonable assumption to make may be seen by referring to the diagrams obtained by Woodward (1959), who examined the motion inside thermals in detail, and her results will later be used to evaluate the constants s and q. (Her figures incidentally give further support to our assumption that external fluid is drawn up the centre, not only in vortex rings in which the buoyant fluid is very concentrated near a ring, but also in buoyant clouds where this assumption is less obviously valid.) The form of (18) also agrees with the solution given in I, as of course it must on purely dimensional grounds.

Equations (16), (17) and (18) may be put into a more convenient non-dimensional form by making the transformations

$$\left.\begin{aligned} t &= \left(\frac{\pi}{qc}\right)^{\frac{1}{2}} G^{-\frac{1}{2}}\tau, \quad F = F_v f, \\ \frac{P}{\rho_1} &= \left(\frac{\pi}{qc}\right)^{\frac{1}{2}} F_v G^{-\frac{1}{2}} p, \quad K = \left(\frac{2s}{qc}\right)^{\frac{1}{2}} F_v^{\frac{1}{2}} k, \\ R &= (2\pi s)^{-\frac{1}{4}} F_v^{\frac{1}{4}} G^{-\frac{1}{4}} r, \quad x = q^{-1}(2\pi s)^{\frac{3}{4}} F_v^{\frac{1}{4}} G^{-\frac{1}{4}} \xi, \end{aligned}\right\} \tag{19}$$

where τ, f, p, k, r and ξ are non-dimensional functions, and F_v is the buoyancy at a virtual origin, to be defined later. The equations become

$$\left.\begin{aligned} p &= kr^2, && (a) \\ \frac{dp}{d\tau} &= f, && (b) \\ v &= \frac{d\xi}{d\tau} = \frac{k}{r}, && (c) \\ \frac{df}{d\tau} &= -r^3\frac{d\xi}{d\tau}, && (d) \\ 2\frac{dk}{d\tau} &= -r^2. && (e) \end{aligned}\right\} \tag{20}$$

From (20) we see that $d^2p/d\tau^2 = -p$; this has the solution $p = p_0 \cos\tau + f_0 \sin\tau$, and hence $f = f_0 \cos\tau - p_0 \sin\tau$, using (20) (*b*).

If the origin of τ is chosen in order to make $p = 0$ and $F = F_v$ (i.e. $f = 1$) there, we have finally

$$\left.\begin{aligned} p &= \sin(\tau + \tau_v) = \sin\tau_1, \\ f &= \cos(\tau + \tau_v) = \cos\tau_1. \end{aligned}\right\} \tag{21}$$

Given f_0 and p_0, this choice may always be made by taking a virtual origin at time $\tau_v = \tan^{-1}(p_0/f_0)$ earlier than the actual origin of time. Note that this (non-dimensional) interval will depend, through (19), on F_0, P_0 and also the density gradient G.

Using (21), we may now obtain easily from (20) the explicit solutions

$$k = [\cos\tau_1 + b]^{\frac{1}{2}}, \tag{22}$$

where b is a constant of integration related to k_v, the value of k at the virtual origin, by

$$k_v^2 = (b+1); \tag{23}$$

also

$$r = \frac{(\sin\tau_1)^{\frac{1}{2}}}{(\cos\tau_1 + b)^{\frac{1}{4}}} \tag{24}$$

and

$$\frac{d\xi}{d\tau} = \frac{(\cos\tau_1 + b)^{\frac{3}{4}}}{(\sin\tau_1)^{\frac{1}{2}}}. \tag{25}$$

The form of solution will depend on the value we assign to b, and the various possibilities will be discussed in turn.

The case $b = 1$

First of all we may note that $b = 1$ corresponds to the situation in which $k = 0$ when $\tau_1 = \pi$, i.e. when $p = 0$; the momentum and the circulation then fall to zero together. When this value of the constant is substituted in (24) and (25), they take the simpler form

$$r = (1 - \cos\tau_1)^{\frac{1}{4}} \tag{24a}$$

and

$$\frac{d\xi}{d\tau} = \frac{\sin\tau_1}{(1 - \cos\tau_1)^{\frac{3}{4}}}. \tag{25a}$$

This last equation may be integrated to give

$$\xi = 4(1 - \cos\tau_1)^{\frac{1}{4}} = 4r. \tag{26}$$

This is precisely the form of solution obtained in I up to the level where the velocity first vanishes, and it clearly leads to a linear increase of radius with distance, as is the case in uniform surroundings. (It is interesting to note, without giving the proof here, that the spread is also linear in an *unstable* linear density gradient, provided the same value of k_v is chosen.) The variation of the other non-dimensional quantities with τ_1 is shown in figure 2.

The case $b > 1$

If $b > 1$, implying that the initial circulation is large enough for k to be non-zero when $p = 0$, the solutions take the form shown in figure 3, which has been calculated for illustration with the particular value $b = 1{\cdot}5$. It is seen that the distance travelled remains finite, although the velocity near the virtual origin and at the maximum height is infinite. This results from the fact that r decreases to zero soon after the buoyancy vanishes, which is shown more clearly in figure 5, where r is plotted against ξ. The spread below this level is again nearly linear.

It is unlikely that the *loss* of fluid from the region moving with the ring can take place in an ordered, smooth manner, and in any case there is an excess of vorticity in the moving region. The obvious physical prediction to be made from figure 3 is that above a certain level, a buoyant vortex ring in stratified surroundings will suddenly collapse and mix with the environment. This effect was in fact noted in the experiments reported in II, but it was there attributed (probably wrongly) to the unstable layering within the structure of the ring, due to the successive addition of layers of fluid of decreasing density.

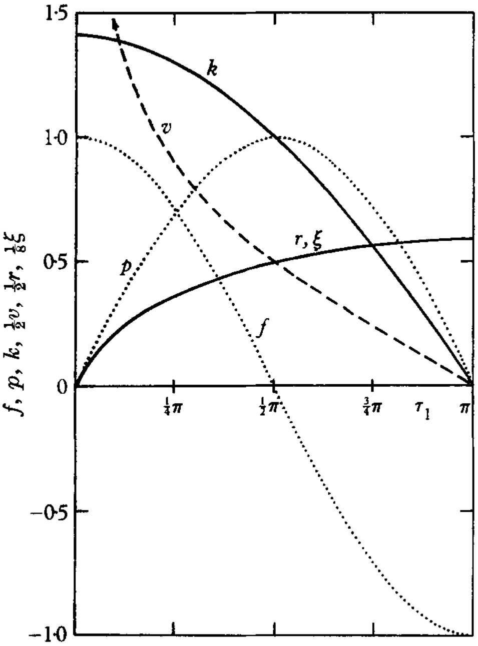

FIGURE 2

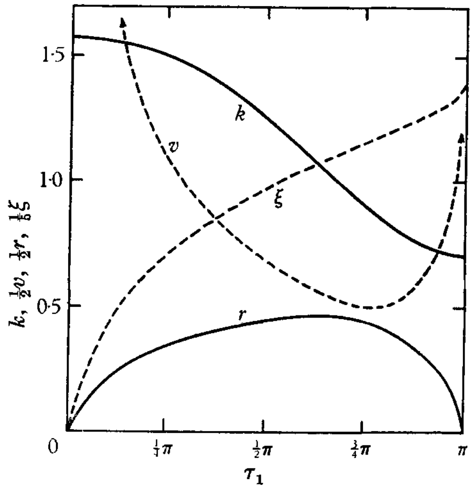

FIGURE 3

FIGURE 2. The non-dimensional solution for a buoyant vortex ring rising through stably stratified surroundings. The initial vorticity is such that the momentum and circulation fall to zero together, i.e. the constant $b = 1$.

FIGURE 3. The non-dimensional solution for a buoyant vortex ring rising through stably stratified surroundings: $b = 1{\cdot}5$, so that the circulation is non-zero when the momentum vanishes.

Note that when b is large, the value of ξ reached before the ring breaks up is, from (25), approximately proportional to $b^{\frac{3}{4}}$, so that using (27) it follows that the maximum height is proportional to $F_v^{-\frac{1}{2}}K_v^{\frac{3}{2}}G^{-\frac{1}{4}}$. This agrees with the prediction made using a dimensional argument in II, and shows again that there can be a considerable increase in the height attained in stable surroundings by a mass having a given buoyancy if it is given extra vorticity (or, equivalently, extra momentum) when it is released. The intermittent release of smoke at a high velocity might be worth considering as a method of increasing the effective height of chimneys, particularly in the very calm, stable conditions when smog is most liable to form; it has been shown on the other hand (Morton 1959) that increasing the momentum of a continuous source under these conditions tends to reduce the final height slightly.

The case $b < 1$

In exactly the same way, when $b = 0{\cdot}5$ (corresponding to the case in which the circulation falls to zero before the momentum does), we can obtain the solutions set out in figure 4. With this assumption, the radius starts increasing linearly, but becomes very large near the maximum height, as is shown in figure 5. The physical reality of this last solution is in some doubt, however, since the least circulation possible for a given buoyancy is that corresponding to the 'buoyant cloud'. In the next section we shall investigate this point further by comparing the buoyant cloud with the condition implied by the 'critical' value of $b = 1$.

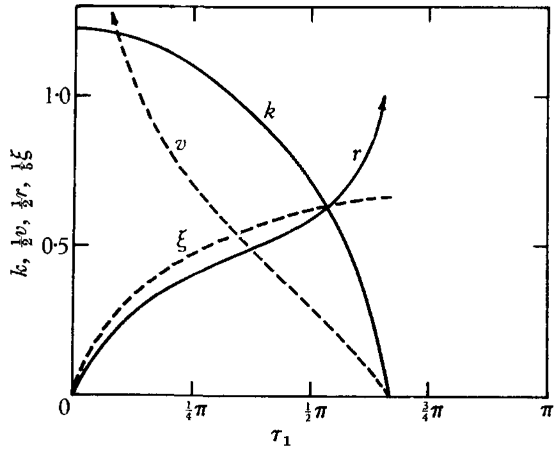

FIGURE 4. The non-dimensional solution for a buoyant vortex ring rising through stably stratified surroundings: $b = 0{\cdot}5$, and the circulation falls to zero before the momentum does.

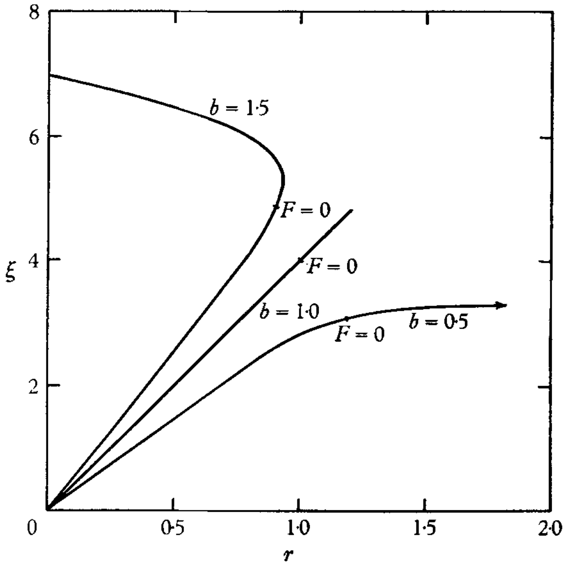

FIGURE 5. Showing the increase of radius with height for buoyant vortex rings in stratified surroundings. The spread is linear for $b = 1$; for $b > 1$ the radius decreases to zero just above the level of zero buoyancy, and for $b < 1$ the radius becomes large at this level.

3.2. *The meaning of the critical condition* $b = 1$

Using (19), equation (23) may be rewritten in the dimensional form

$$\frac{K_v^2}{F_v} = \left(\frac{2s}{qc}\right)(b+1). \tag{27}$$

Putting $b = 1$, and using (5), the half-angle of spread corresponding to the critical case is seen to be

$$\alpha_c = \frac{q}{8\pi s}. \tag{28}$$

Thus, we must now make numerical estimates of the geometrical factors q and s. The shapes and streamlines within thermals which Woodward (1959) has published immediately allow us to evaluate the volume in terms of R as we have defined it, and hence show that q is about 13. In order to obtain a precise value for s, we would need to know the variation of density through the centre of the thermal and in its wake; but by examining Woodward's diagrams showing the distortion of initially horizontal layers in a uniform ambient fluid, we can say that s must be about 3.

Inserting these constants in (28) gives the value $\alpha_c = 0{\cdot}17$. This is very close to the half-angle of spread for the buoyant cloud which was discussed in §2.3, and it suggests that it is *not* possible to realize physically the solution for $b < 1$. The agreement between the two values is in fact so close that it is tempting to seek a physical reason why they might be identical. It is difficult, however, to convince oneself that there is an exact parallel between the processes of generation of vorticity in a cloud starting from rest in uniform surroundings and the destruction of vorticity as the cloud is brought to rest in a density gradient. It is worth remarking that *dimensionally* the two problems are the same, since the density gradient does not enter into the condition specifying the critical value of b except through the (small) differences between the values of K and F at the real and virtual sources. This is not so for the pair vortex, as will be shown in the next section.

3.3. *Buoyant vortex pairs*

The development of the theory for a vortex pair will proceed in a parallel form to that given above for vortex rings, and with the same assumptions except of course for the obvious modifications necessary in two dimensions. For conciseness we will proceed directly to the non-dimensional set of equations in the variables τ, f', p', k, r and ξ which are related to the physical variables by

$$\left.\begin{aligned} t &= \left(\frac{8\pi}{q'}\right)^{\frac{1}{2}} G^{-\frac{1}{2}}\tau, \quad F' = F'_v f', \\ \frac{P'}{\rho_1} &= \left(\frac{8\pi}{q'}\right)^{\frac{1}{2}} F'_v G^{-\frac{1}{2}} p', \quad K = \left(\frac{8\pi}{q'}\right)^{\frac{1}{2}} \left(\frac{3s'}{8}\right)^{\frac{1}{3}} F'^{\frac{2}{3}}_v G^{-\frac{1}{6}} k, \\ R &= (3s')^{-\frac{1}{3}} F'^{\frac{1}{3}}_v G^{-\frac{1}{3}} r, \quad x = (3s')^{\frac{2}{3}} (q')^{-1} F'^{\frac{1}{3}}_v G^{-\frac{1}{3}} \xi. \end{aligned}\right\} \tag{29}$$

Here q' and s' are new constants corresponding to q and s for the vortex ring. The governing equations become

$$\left.\begin{aligned} p' = kr, \quad \frac{dp'}{d\tau} = f', \quad \frac{d\xi}{d\tau} = \frac{k}{r}, \\ \frac{df'}{d\tau} = -r^2\frac{d\xi}{d\tau}, \quad \frac{3}{2}\frac{dk}{d\tau} = -r^2. \end{aligned}\right\} \tag{30}$$

The solution for p' and f' may be obtained in a form similar to that for the vortex ring

$$p' = \sin(\tau+\tau_v) = \sin\tau_1, \quad f' = \cos(\tau+\tau_v) = \cos\tau_1, \tag{31}$$

where the virtual origin of τ has been chosen at a time $\tau_v = \tan^{-1}(p'_0/f'_0)$ earlier than the actual origin, in order to make $p' = 0$ when $f' = 1$. This time will again depend on G as well as on F_0 and P_0.

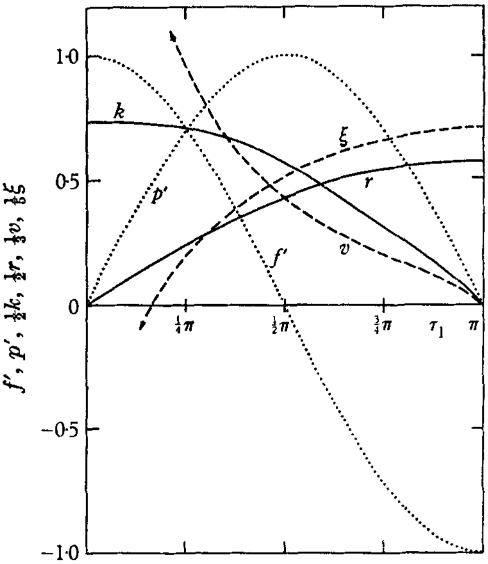

FIGURE 6. The non-dimensional solution for a buoyant vortex pair in a stably stratified fluid: $k_v^3 = \pi$, so that the circulation and momentum fall to zero together.

Using (31) in (30), k may now be expressed as

$$k = (\sin\tau_1\cos\tau_1 - \tau_1 + k_v^3)^{\frac{1}{3}}, \tag{32}$$

where k_v is the value of k at the virtual origin. Also

$$r = \frac{\sin\tau_1}{(\sin\tau_1\cos\tau_1 - \tau_1 + k_v^3)^{\frac{1}{3}}} \tag{33}$$

and

$$\frac{d\xi}{d\tau} = \frac{(\sin\tau_1\cos\tau_1 - \tau_1 + k_v^3)^{\frac{2}{3}}}{\sin\tau_1}. \tag{34}$$

It is again convenient to distinguish a 'critical' case, such that $k = 0$ when $p = 0$, or when $\tau_1 = \pi$. This occurs when $k_v^3 = \pi$; and the solution with this value of the constant is shown in figure 6. The integration of (34), which has been carried out numerically, must be started at an arbitrary value of τ_1, since the virtual origin is at $\xi = -\infty$ (as it is for a pair vortex in a uniform ambient fluid). The pair, however, comes to rest at a finite height above the real origin, with a finite radius, as is shown more clearly in figure 9.

As examples of the other forms of solution possible, we have taken the cases $k_v^3 = 4$ and $k_v^3 = 2$. When $k_v^3 > \pi$, and there is excess vorticity in the pair, the solution (figure 7) indicates that the radius begins to decrease soon after the buoyancy vanishes; the velocity and distance travelled become infinite (in contrast to the corresponding vortex ring solution which gave zero radius at a finite height). Again we would suggest that physically this implies that such a buoyant pair would become unstable and suddenly mix with its surroundings at a certain height.

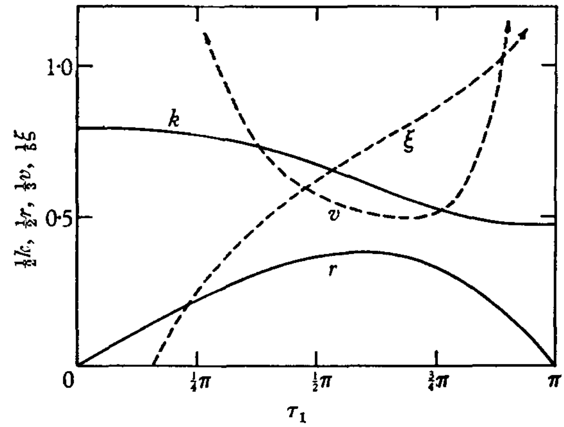

FIGURE 7. The non-dimensional solution for a buoyant vortex pair in a stably stratified fluid: $k_v^3 = 4$.

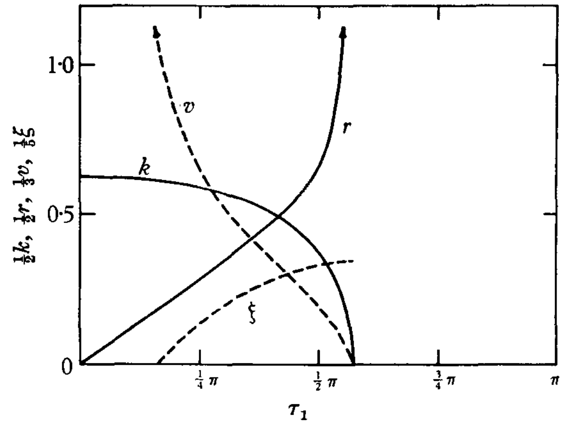

FIGURE 8. The non-dimensional solution for a buoyant vortex pair in a stably stratified fluid: $k_v^3 = 2$.

When k_v^3 is less than the critical value π, the radius becomes large at a finite height (figure 8). The different forms of the relation between radius and height are again shown in figure 9.

It should be noted that there is an important difference between the 'critical' conditions for vortex rings and vortex pairs. We saw in the last section that it is meaningful to look for at least a numerical relation between the conditions in a buoyant cloud and the critical state, since these both depend only on K_v and F_v which will be little different from K_0 and F_0. For the pair vortex, however, the distribution of vorticity established through the action of buoyancy forces depends also on a length scale (equation (14)), whereas the critical condition depends explicitly on the density gradient in the surroundings (equation (29)).

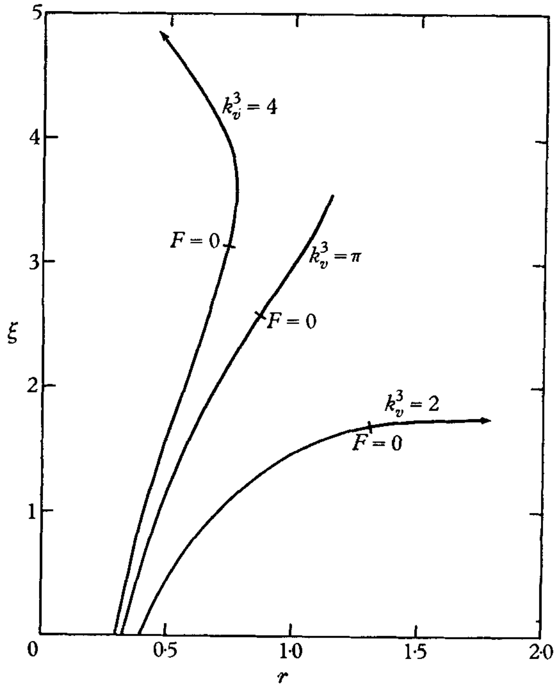

FIGURE 9. The increase of radius of buoyant vortex pairs in stably stratified surroundings as a function of height, starting from the height corresponding to $r_1 = 0{\cdot}5$. When $k_v^3 = \pi$, the pair comes to rest at a finite height and radius; when $k_v^3 > \pi$, the radius decreases soon after the buoyancy vanishes, but only becomes zero at an infinite height; and when $k_v^3 < \pi$, the radius becomes infinite near the level of zero buoyancy.

Even if experiments were available to allow us to evaluate q' and s' (and none have so far been reported), the size and the density gradient would have to be known before the appropriate solution could be applied. It seems possible in this case that all three types of solution may correspond to physical reality under some atmospheric and initial conditions.

REFERENCES

LAMB, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.

MORTON, B. R., TAYLOR, G. I. & TURNER, J. S. 1956 *Proc. Roy. Soc.* A, **234**, 1.

MORTON, B. R. 1959 *J. Fluid Mech.* **5**, 151.

ROUSE, H., YIH, C. S. & HUMPHREYS, H. W. 1952 *Tellus*, **4**, 201.

SCORER, R. S. 1958 *Natural Aerodynamics*. London: Pergamon Press.

TURNER, J. S. 1957 *Proc. Roy. Soc.* A, **239**, 61.

WOODWARD, B. 1959 *Quart. J. R. Met. Soc.* **85**, 144.

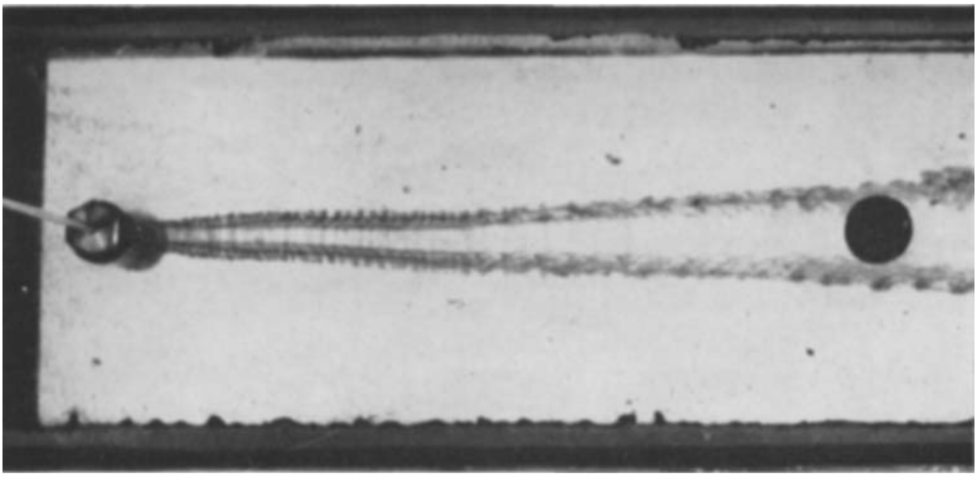

FIGURE 1 (plate 1). Photograph showing a plan view of a bent-over, buoyant plume. Note how it splits sideways into two line vortices, which spread nearly linearly with distance. The photograph was taken through the larger face of a 10 cm × 5 cm water channel; the bolt heads are spaced at 20 cm intervals. The water velocity was about 3 cm/sec, and the rate of release of buoyant fluid through a 1·5 mm tube was adjusted so that the plume bent over near the centre of the channel.